Stability and phase coherence of trapped 1D Bose gases

D.M. Gangardt[1] and G.V. Shlyapnikov[1,2,3]

[1]*Laboratoire Kastler-Brossel, Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris, France*
[2]*FOM Institute for Atomic and Molecular Physics, Kruislaan 407, 1098 SJ Amsterdam, The Netherlands*
[3]*Russian Research Center Kurchatov Institute, Kurchatov Square, 123182 Moscow, Russia*

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We discuss stability and phase coherence of 1D trapped Bose gases and find that inelastic decay processes, such as three-body recombination, are suppressed in the strongly interacting (Tonks-Girardeau) and intermediate regimes. This is promising for achieving these regimes with a large number of particles. "Fermionization" of the system reduces the phase coherence length, and at $T = 0$ the gas is fully phase coherent only deeply in the weakly interacting (Gross-Pitaevskii) regime.

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Recent success in creating quantum degenerate 1D trapped atomic gases [1, 2, 3] has stimulated an interest in correlation properties of these systems. The 1D regime is reached by tightly confining the radial motion of atoms in a cylindrical trap to zero point oscillations. The difference between such a kinematically 1D gas and a purely 1D gas is related only to the value of the interparticle interaction, which depends on the radial confinement.

The 1D Bose gas with repulsive short-range interactions characterized by the coupling constant $g > 0$ exhibits remarkable properties. Counterintuitively, it becomes more non-ideal with decreasing 1D density n [4, 5]. The equation of state and correlation functions depend crucially on the parameter

$$\gamma = mg/\hbar^2 n, \tag{1}$$

where m is the atom mass. For comparatively large n, the parameter $\gamma \ll 1$ and $\sqrt{\gamma}$ represents the ratio of the mean interparticle separation $1/n$ to the correlation length $l_c = \hbar/\sqrt{mng}$. In this case the gas is in the weakly interacting or Gross-Pitaevskii (GP) regime, and the amplitude of the boson field obeys the familiar Gross-Pitaevskii equation. For sufficiently low densities (or large g), one has $\gamma \gg 1$ and the strongly interacting or Tonks-Girardeau (TG) regime is reached. In this regime the 1D gas acquires fermionic properties in the sense that the wave function strongly decreases as particles approach each other (see [4, 5] and cf. [6]). For $\gamma = \infty$ any correlation function of the density can be calculated straightforwardly [7] by using the exact mapping onto the system of free fermions found by Girardeau [4].

A uniform 1D system of bosons interacting via a delta-functional potential is integrable by using the Bethe ansatz and has been a subject of extensive theoretical studies. Lieb and Liniger [5] have calculated the ground state energy and excitation spectrum for any value of the parameter γ. Yang and Yang [8] have studied thermodynamic properties of this system at finite temperatures and found no phase transition for $T > 0$. The absence of long-range order and true Bose-Einstein condensate in finite-temperature 1D Bose gases follows from the Bogoliubov k^{-2} theorem as has been expounded in [9]. The long-range order is destroyed by long-wave fluctuations of the phase leading to an exponential decay of the one-particle density matrix at large distances [10, 11]. A similar picture is found at $T = 0$ [12], where the density matrix undergoes a power-law decay [11, 13, 14]. In the past few decades, a general approach has been developed for exact calculations of one- and two-particle correlation functions at an arbitrary γ (see [15, 16] for review).

Realization of 1D trapped Bose gases raises the question of their phase coherence and stability. The phase coherence properties are strongly influenced by the trapping potential, which introduces a finite size of the system and provides a low-momentum cut-off of the phase fluctuations. In the GP regime at sufficiently low T, the phase fluctuations are suppressed and the equilibrium state is a true condensate. At higher temperatures, it is transformed into a phase-fluctuating condensate [17]. The dynamics, interference effects, and excitations of 1D trapped Bose gases are currently a subject of active studies [18, 19, 20, 21]. Of particular interest is the change in phase coherence properties while going from the GP to the TG regime, where the phase coherence is completely lost.

The strong transverse confinement required for the 1D regime can lead to high 3D densities of a trapped gas. At a large number of particles, the 3D density can exceed 10^{15} cm^{-3}, and one expects a fast decay due to three-body recombination. It is then crucial to understand how the correlation properties of the gas influence the decay rate.

In this Letter, we discuss stability of 1D Bose gases and calculate local density correlators, as those are responsible for inelastic decay processes [22]. We find that the decay rates are suppressed in the TG and intermediate regimes, which is promising for achieving these regimes with a large number of particles. We then analyze phase coherence of a trapped 1D Bose gas and show that vacuum fluctuations of the phase make the zero-temperature coherence length smaller than the Thomas-Fermi size of

the sample, unless the gas is deeply in the GP regime.

The 1D regime in a trapped gas is realized if the amplitude of transverse zero point oscillations $l_0 = \sqrt{\hbar/m\omega_0}$ is much smaller than the longitudinal correlation length $l_c = \hbar/\sqrt{m\mu}$, where ω_0 is the frequency of the transverse confinement and the chemical potential of the 1D system is $\mu \ll \hbar\omega_0$. One then has a 1D system of bosons interacting with each other via a short-range potential characterized by an effective coupling constant $g > 0$. This constant is expressed through the 3D scattering length a [23], assuming that l_0 greatly exceeds the radius of interaction between atoms. For a positive $a \ll l_0$ we have

$$g = 2\hbar^2 a/ml_0^2, \tag{2}$$

and a characteristic distance $\hbar^2/mg$ related to the interaction between particles in the described 1D problem is $\sim l_0^2/a \gg l_0$. In the GP regime, the chemical potential $\mu \approx gn$, and the condition $l_c \gg l_0$ leads to the inequality $na \ll 1$. In the TG regime the correlation length $l_c \sim 1/n$, and one should have $nl_0 \ll 1$. We thus see that at any value of γ it is sufficient to satisfy the inequalities $a \ll l_0 \ll 1/n$. Then the 1D regime is reached and the system can be analyzed on the basis of the 1D Lieb-Liniger model, assuming a delta-functional interatomic potential with the coupling constant g given by Eq. (2).

The rate of three-body recombination is proportional to the local three-particle correlation function $g_3 = \langle\Psi^\dagger(x)\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\Psi(x)\rangle$ [22], where $\Psi(x)$ is the field operator of atoms and the symbol $\langle\ldots\rangle$ denotes the expectation value. Similarly, the rates of two-body inelastic processes involve the correlation function $g_2 = \langle\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x)\Psi(x)\rangle$. Assuming that local correlation properties are insensitive to the geometry of the system we consider a uniform 1D gas of N bosons on a ring of circumference L. The Hamiltonian of the system reads:

$$H = \int dx\left[(\hbar^2/2m)\partial_x\Psi^\dagger\partial_x\Psi + (g/2)\Psi^\dagger\Psi^\dagger\Psi\Psi\right]. \tag{3}$$

For finding g_2 at $T = 0$, we use the Hellmann-Feynman theorem [24, 25]. This is similar to the calculation of the mean interaction energy in ref. [5]. Namely, one shows that the expectation value of the four-operator term in the Hamiltonian (3) is proportional to the derivative of the ground state energy with respect to the coupling constant: $dE_0/dg = \langle\Phi_0|dH/dg|\Phi_0\rangle = g_2L/2$. The first identity follows from the normalization of the ground state wave function Φ_0, and the second one is obtained straightforwardly from the Hamiltonian (3). The ground state energy can be written as $E_0 = Ne(\gamma)\hbar^2n^2/2m$, where the quantity $e(\gamma)$ is a solution of the Lieb-Liniger equations [5] and is calculated numerically for any value of γ [19, 26]. For the two-particle local correlation function we then obtain $g_2(\gamma) = n^2 de(\gamma)/d\gamma$. The function $g_2(\gamma)/n^2$ is shown in Fig. 1. For small values of γ, we obtain numerically the result which coincides with that

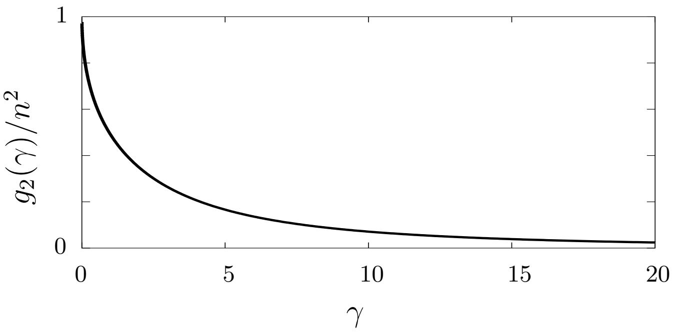

FIG. 1: Local correlation function g_2 versus γ.

following from the Bogoliubov approach:

$$g_2(\gamma)/n^2 = 1 - 2\sqrt{\gamma}/\pi, \quad \gamma \ll 1. \tag{4}$$

In the limit of large γ, we have $e(\gamma) = \left(\pi^2/3\right)(1 - 4/\gamma)$, and the two-particle correlation function is given by

$$g_2(\gamma)/n^2 = 4\pi^2/3\gamma^2, \quad \gamma \gg 1. \tag{5}$$

The results in Fig. 1 and Eq. (5) clearly show that two-particle correlations and, hence, the rates of pair inelastic processes, are suppressed for $\gamma \gtrsim 1$. This provides a possibility for identifying the TG and intermediate regimes of a trapped 1D Bose gas through the measurement of photoassociation in pair interatomic collisions.

The three-particle local density correlator g_3 cannot be obtained from the Hellmann-Feynman theorem. In the weak coupling GP regime ($\gamma \ll 1$) one can use the Bogoliubov approach, which immediately yields

$$g_3(\gamma)/n^3 \simeq 1 - 6\sqrt{\gamma}/\pi, \quad \gamma \ll 1. \tag{6}$$

For the TG regime ($\gamma \gg 1$), we have developed a method for calculating the leading behavior of local density correlators. Details will be given elsewhere, and here we present a compact derivation of g_3 at $T = 0$. In first quantization the expression for this function reads

$$g_3(\gamma) = \frac{N!}{3!(N-3)!}\int dX\left|\Phi_0^{(\gamma)}(0,0,0,x_4,\ldots,x_N)\right|^2, \tag{7}$$

where $dX = dx_4\ldots dx_N$, and $\Phi_0^{(\gamma)}$ is the ground state wave function given in the domain $0 < x_1 < \ldots < x_N < L$ by the Bethe ansatz solution:

$$\Phi_0^{(\gamma)}(x_1,x_2,\ldots,x_N) \propto \sum_P a(P)\exp\left\{i\sum k_{P_j}x_j\right\}, \tag{8}$$

where P is a permutation of N numbers, quasimomenta k_j are solutions of the Bethe ansatz equations, and

$$a(P) = \prod_{j<l}\left(\frac{i\gamma n + k_{P_j} - k_{P_l}}{i\gamma n - k_{P_j} + k_{P_l}}\right)^{\frac{1}{2}}.$$

For $\gamma \gg 1$, we extract the leading contribution to $\Phi_0^{(\gamma)}$ at three coinciding points by symmetrizing the amplitudes

$a(P)$ over the first three elements of the permutation P:

$$\frac{1}{3!}\sum_{p} a(P_p) \simeq \frac{\epsilon_P}{(i\gamma n)^3}\prod_{j<l}(k_{P_j}-k_{P_l}), \tag{9}$$

where $P_p = P_{p_1}, P_{p_2}, P_{p_3}, P_4, \ldots P_N$, and $j,l = 1,2,3$. The sign of the permutation P is ϵ_P, and p runs over six permutations of $1,2,3$. For large γ, the difference of quasi-momenta k_j from their values at $\gamma=\infty$ is of order $1/\gamma$ and can be neglected. Then, from Eqs. (8) and (9), we conclude that to this level of accuracy the ground state wave function at three coinciding points is given by derivatives of the wave function of free fermions $\Phi_0^{(\infty)}(x_1,x_2,x_3,x_4,\ldots)$ at $x_1=x_2=x_3=0$:

$$\Phi_0^{(\gamma)}(0,0,0,x_4,\ldots) \simeq -\frac{1}{(\gamma n)^3}\Big[\prod_{j<l}(\partial_{x_j}-\partial_{x_l})\Big]\Phi_0^{(\infty)}. \tag{10}$$

Substituting Eq. (10) into Eq. (7) we express the local correlator g_3 through derivatives of the three-body correlation function of free fermions. Using Wick's theorem, the latter is given by a sum of products of one-particle fermionic Green's functions $G(x-y) = \int_{-k_F}^{k_F} dk\, e^{ik(x-y)}/2\pi$, where $k_F = \pi n$ is the Fermi wavevector. The calculation from Eq. (7) is then straightforward and we obtain ($\gamma \gg 1$)

$$\frac{g_3(\gamma)}{n^3} = \frac{36}{\gamma^6 n^9}\Big[(G'')^3 - G^{(4)}G''G\Big] = \frac{16\pi^6}{15\gamma^6}, \tag{11}$$

where G and its derivatives are taken at $x-y=0$.

In fact, in our problem the correlation functions g_2 and g_3 are slightly nonlocal. They are related to interparticle distances $r \sim l_0 \ll l_c$, since at smaller r the relative motion of particles is three-dimensional and the local correlators do not change. For large γ, at distances $\sim l_0$ an extra (coordinate-dependent) contribution to g_2 is of the order of $(nl_0)^2$ and to g_3 of the order of $(nl_0)^6$ (see, e.g. [7] and references therein). Under the condition $l_0 \gg a$, these contributions can be neglected as they are much smaller than the results of Eqs. (5) and (11), respectively.

Our method is readily generalized for the case of finite temperature by considering the temperature-dependent Green's functions of free fermions. For $T \ll \mu$ we obtain a small correction $\sim (T/\mu)^2$ to the zero-temperature result. The same conclusion holds for the GP regime ($\gamma \ll 1$).

Thus, from Eq. (11) we conclude that the three-body decay of 1D trapped Bose gases is strongly suppressed in the TG regime. Moreover, Eq. (6) shows that even in the GP regime with $\gamma \approx 10^{-2}$, one has a 20% reduction of the three-body rate. Thus, one also expects a significant reduction of the three-body decay in the intermediate regime.

For $l_0 \gg a$, the recombination process in 1D trapped gases occurs at interparticle distances much smaller than l_0. Therefore, the equation for the recombination rate is the same as in 3D cylindrical Bose-Einstein condensates with the Gaussian radial density profile. There is only an extra reduction by a factor of g_3/n^3. A characteristic decay time τ is then given by the relation $\tau^{-1} = \alpha_{3D} n_{3D}^2 (g_3/3n^3)$, where α_{3D} is the recombination rate constant for a 3D condensate, and $n_{3D} = n/(\pi l_0^2)$ is the maximum 3D density. Even for $n_{3D} \sim 10^{15}$ cm^{-3}, the lifetime τ can greatly exceed seconds when approaching the TG regime. For example, this is the case for ^{87}Rb ($\alpha_{3D} \sim 10^{-29}$ cm^6/s) optically trapped with $\omega_0 \approx 100$ kHz ($l_0 \approx 200$ Å), assuming $L \approx 100$ μm and $N = 200$. Then one has $\gamma \approx 10$ and Eq. (11) predicts a reduction of the three-body rate by more than three orders of magnitude.

We now turn to phase coherence of a 1D Bose gas in a harmonic potential $V(x) = m\omega^2 x^2/2$. We consider the case of $T=0$ and rely on the hydrodynamic approach [14] in which long-wave properties of the 1D fluid are described in terms of two conjugate fields, density fluctuations δn and phase ϕ. They satisfy the commutation relation $[\delta n(x), \exp(i\phi(x'))] = \delta(x-x')\exp(i\phi(x))$. We assume the Thomas-Fermi regime and use the local density approximation [19, 20]: the mean density $n(x)$ is obtained from the local equation of state $\mu(n(x)) = \mu_0 - V(x)$, where $\mu(n)$ is the chemical potential for the Lieb-Liniger problem. The density is non-zero only within the Thomas-Fermi radius $R_{TF} = \sqrt{2\mu_0/m\omega^2}$, and the normalization condition $\int_{-R_{TF}}^{R_{TF}} n(x)dx = N$ gives a relation between μ_0 and N. Equations of motion for the fields δn and ϕ follow from the quantum Hamiltonian:

$$H_q = \frac{\hbar}{2\pi}\int dx\,\left(v_N(\pi\delta n)^2 + v_J(\partial_x\phi)^2\right) = \hbar\sum_j \Omega_j {b^\dagger}_j b_j,$$

where Ω_j and b_j are frequencies and annihilation operators of elementary excitations. The quantities $v_N(x) = (\pi\hbar)^{-1}\partial\mu/\partial n$ and $v_J(x) = \pi\hbar n(x)/m$ determine the local sound velocity $\sqrt{v_N(x)v_J(x)}$ and the local Luttinger parameter $K(x) = \sqrt{v_J(x)/v_N(x)}$. The Hamiltonian H_q is a generalization of the effective harmonic Hamiltonian of Ref. [14] to a non-uniform system.

Using the density-phase representation for the field operators, we calculate the one-particle density matrix $g_1(x,x') = \langle \Psi^\dagger(x)\Psi(x')\rangle$ for $|x-x'| \gg l_c$. As the density fluctuations are small, this matrix reduces to

$$g_1(x,x') = \sqrt{n(x)n(x')}\exp\{-\langle(\phi(x)-\phi(x'))^2\rangle/2\}.$$

The phase operator is given by its expansion in eigenmodes labeled by an integer quantum number $j>0$:

$$\phi(x) = -i\sum_j \left(\frac{\pi v_N(0)}{2\Omega_j R_{TF}}\right)^{1/2} f_j(y)b_j + \text{H.c.}, \tag{12}$$

where we have introduced a dimensionless coordinate $y = x/R_{TF}$. The eigenfunctions f_j are normalized by the

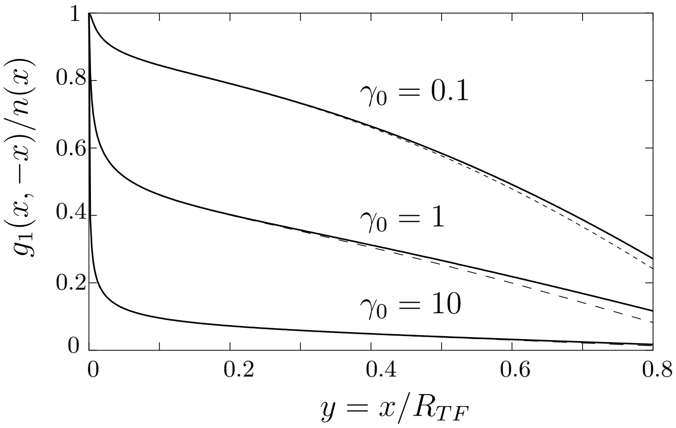

FIG. 2: The density matrix $g_1(x,-x)$ for $N = 10^4$ and various values of γ_0. The solid curves show numerical results, and the dashed curves the results of the quasiclassical approach.

condition $\int_{-1}^{1} dy\,(v_N(0)/v_N(y))\, f_j^*(y) f_{j'}(y) = \delta_{jj'}$. From the Hamiltonian H_q we obtain the continuity and Euler equations which lead to the eigenmode equation,

$$(1-y^2)f_j'' - (2y/\beta(y))f_j' + (2\Omega^2/\beta(y)\omega^2)f_j = 0. \quad (13)$$

The quantity $\beta(y) = d\ln\mu/d\ln n$ is determined by the local parameter $\gamma(x) = mg/\hbar^2 n(x)$. In the TG regime, we have $\beta = 2$, and in the GP regime $\beta = 1$. The coordinate dependence of β is smooth, and we simplify Eq. (13) by setting $\beta(y) = \beta_0$, where β_0 is the value of β at maximum density. This simplification has been used [20, 27] to study the excitation spectrum of trapped 1D Bose gases. Then Eq. (13) yields the spectrum $\Omega_j^2 = \omega^2(j\beta_0/2)(j + 2/\beta_0 - 1)$, and the eigenfunctions $f_j(y)$ are Jacobi polynomials $P_j^{(\alpha,\alpha)}(y)$ with $\alpha = 1/\beta_0 - 1$.

Using Eq. (12), the mean square fluctuations of the phase $\langle(\phi(x) - \phi(x'))^2\rangle$ are reduced to the sum over j-dependent terms containing eigenfunctions f_j and eigenfrequencies Ω_j. For the vacuum phase fluctuations this sum is logarithmically divergent at large j, which is similar to the high-momentum divergence in the uniform case. Accordingly, we introduce a cut-off j_{max} following from the condition $\hbar\Omega_j \approx \min\{\mu(x), \mu(x')\}$ and ensuring a phonon-like character of excitations at distances x and x'. The vacuum phase fluctuations have been calculated by using two approaches: numerical summation over the eigenmodes with f_j, Ω_j from the simplified Eq. (13), and quasiclassical approach assuming that the main contribution comes from excitations with $j \gg 1$. In the latter case, for $x' = -x$ we obtain $\langle(\phi(x) - \phi(-x))^2\rangle \approx K^{-1}(x)\ln\{|2x|/l_c(x)\}$, which is close to Haldane's result for a uniform system [14] with the Luttinger parameter $K(x)$ and correlation length $l_c(x)$.

The dependence of g_1 on the dimensionless coordinate y is governed by two parameters: $\gamma_0 \equiv \gamma(0)$ and the number of particles N. In Fig. 2, we present the quantity $g_1(y,-y)/n(y)$ for $N = 10^4$ and various values of γ_0. As expected, the phase coherence is completely lost in the TG regime ($\gamma_0 \gg 1$). Moreover, on a distance scale $x \sim R_{TF}$ the coherence is already lost for $\gamma_0 \approx 1$. Thermal fluctuations of the phase are readily included in our scheme and will be discussed elsewhere.

In conclusion, we have found an enhanced stability of a trapped 1D Bose gas in the Tonks-Girardeau and intermediate regimes and described the reduction of phase coherence in these regimes.

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